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THE BARNETT APPROXIMATION OF THE DISTRIBUTION FUNCTION AND THE SUPER-BARNETT CONTRIBUTIONS TO THE STRESS TENSOR AND THE HEAT FLUX

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Exact solution of the equation that determines the Barnett correction to the velocity distribution function is obtained for Maxwellian molecules. The solution contains a number of new terms that are absent in Barnett's calculations. The derived distribution function is used for calculating super-Barnett terms in the stress tensor and in the heat flux. Errors which appeared in [1] in the calculation of these terms by another method have been corrected.

1. The Barnett correction to the distribution function. In the case of Maxwellian molecules the integral equation which is satisfied by $f^{(1)}$ can be exactly solved [2]. The equation that determines $f^{(2)} = f^{(0)}\Phi^{(2)}$ can be solved in the same manner. It is of the form

$$\frac{\partial_1}{\partial t} f^{(0)} + \left(\frac{\partial_0}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) f^{(1)} - I(f^{(1)}, f^{(1)}) = L(\Phi^{(2)}) \tag{1.1}$$

with additional conditions [2]

$$\int \psi(\mathbf{v}) f^{(2)} d\mathbf{v} = 0, \quad \psi(\mathbf{v}) = 1, \mathbf{v}, (\mathbf{v}-\mathbf{u})^2 \tag{1.2}$$

where

$$f^{(1)} = f^{(0)}\left(A_\alpha \frac{1}{T}\frac{\partial T}{\partial r_\alpha} + B_{\alpha\beta}e_{\alpha\beta}\right) =$$

$$f^{(0)}\left[\frac{3\mu}{2pT}\frac{\partial T}{\partial r_\alpha} V_\alpha\left(\frac{5}{2} - W^2\right) - \frac{\rho\mu}{p^2} e_{\alpha\beta}V_\alpha V_\beta\right]$$

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha}\right) - \frac{1}{3}\delta_{\alpha\beta}\operatorname{div}\mathbf{u}, \quad W^2 = \frac{mV^2}{2kT}, \quad \mathbf{V} = \mathbf{v} - \mathbf{u}$$

and L is the linearized integral of collisions. The remaining notation conforms to [2].

We shall indicate only the basic aspects of solution, omitting detailed calculations.

Equation (1.1) is an inhomogeneous linear integral Fredholm equation of the second kind. The conditions of its solvability (orthogonality of the left-hand side of (1.1) with respect to solutions of the homogeneous equation) are defined by formulas

$$\frac{\partial_1 n}{\partial t} = 0, \quad \frac{\partial_1 u_\alpha}{\partial t} = -\frac{1}{\rho}\frac{\partial p_{\alpha\beta}^{(1)}}{\partial r_\beta} \tag{1.3}$$

$$\frac{3}{2} nk\frac{\partial_1 T}{\partial t} = -\operatorname{div}\mathbf{q}^{(1)} - p_{\alpha\beta}^{(1)}\frac{\partial u_\alpha}{\partial r_\beta}$$

We then represent the left-hand side of (1.1) in the form of a linear combination of eigenfunctions of the linearized collision integral.

The most complicated is the calculation of the collision integral I. Using the method expounded in [1] we obtain

$$I(A_\alpha, A_\beta) = f^{(0)} \frac{9}{8}\left(1 - \frac{1}{2}\frac{A_4}{A_2}\right)\frac{\mu}{p}\langle V_\alpha V_\beta\rangle S^{(2)}_{5/2} \tag{1.4}$$

$$I(B_{\alpha\beta}, B_{\gamma\delta}) = f^{(0)}\left[\frac{9}{16}\left(1 - \frac{35}{27}\frac{A_4}{A_2}\right)\frac{\rho^2\mu}{p^3}\langle V_\alpha V_\beta V_\gamma V_\delta\rangle + \frac{4}{21}\frac{\rho\mu}{p^2}\delta_{\beta\gamma}\langle V_\alpha V_\delta\rangle S^{(1)}_{5/2} - \frac{8}{45}\frac{\mu}{p}\delta_{\alpha\delta}\delta_{\beta\gamma}S^{(2)}_{1/2}\right]$$

$$I(B_{\alpha\beta}, A_\gamma) + I(A_\gamma, B_{\alpha\beta}) = -f^{(0)}\left[\frac{1}{2}\left(1 - \frac{5}{8}\frac{A_4}{A_2}\right)\frac{\rho\mu}{p^2}\langle V_\alpha V_\beta V_\gamma\rangle S^{(1)}_{7/2} + \frac{1}{5}\frac{\mu}{p}\delta_{\beta\gamma}V_\alpha S^{(2)}_{3/2}\right]$$

where angle brackets denote the irreducible symmetric part of the tensor [3] and $S^{(r)}_{l+1/2}(W^2)$ are Sonin polynomials (in (1.4) and subsequently the argument W^2 is omitted for brevity).

Functions $\psi_{rl} = \langle V_{\alpha_1} V_{\alpha_2} \ldots V_{\alpha_l}\rangle S^{(r)}_{l+1/2}$ are eigenfunctions of the collision operator (λ_{rl} are eigenvalues) [3]

$$L(\psi_{rl}) = -f^{(0)}\lambda_{rl}\psi_{rl} \tag{1.5}$$

The remaining terms in the left-hand side of (1.1) after differentiation in (1.3) can also be expressed in terms of eigenfunctions of operator L.

We seek a particular solution of Eq. (1.3) of the form

$$\Phi^{(2)} = \sum_{l=0}^{4}\sum_{s}\sum_{r=0}^{\infty} R^{(s)}_{rl}(\mathbf{r}, t)\,\Gamma^{(s)}_{\alpha_1\alpha_2\ldots\alpha_l}\psi_{rl}$$

where $\Gamma^{(s)}_{\alpha_1\alpha_2\ldots\alpha_l}$ are tensors of rank l that appear in the left-hand side of (1.1) and consist of derivatives of macroscopic parameters of gas. Using (1.5) and the orthogonality of Sonin polynomials, it is possible to determine in the conventional manner the unknown coefficients $R_{rl}^{(s)}$ [3].

As the result, we obtain for the Barnett correction of the distribution function the following expression:

$$\Phi^{(2)} = \frac{\mu^2}{\rho p}\left\{\frac{4\rho}{3p}e_{\alpha\beta}e_{\beta\alpha}S^{(2)}_{1/2} + \frac{3}{T}\frac{\partial^2 T}{\partial r_\alpha^2}S^{(2)}_{1/2} + \frac{3}{T^2}\frac{\partial T}{\partial r_\alpha}\frac{\partial T}{\partial r_\alpha}\left[\left(\frac{7}{2} + \frac{T}{\mu}\frac{d\mu}{dT}\right)S^{(2)}_{1/2} - 2S^{(3)}_{1/2}\right] - \frac{3}{pT}\frac{\partial T}{\partial r_\alpha}\frac{\partial p}{\partial r_\alpha}S^{(2)}_{1/2}\right\} + V_\alpha\frac{\mu^2}{p^2}\left\{-\frac{3}{2T}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{\partial T}{\partial r_\alpha}\operatorname{div}\mathbf{u}S^{(1)}_{3/2} - \frac{9}{4T}\left(\frac{D_0}{Dt}\frac{\partial T}{\partial r_\alpha} - \frac{\partial u_\beta}{\partial r_\alpha}\frac{\partial T}{\partial r_\beta}\right)S^{(1)}_{3/2} - \right. \tag{1.6}$$

$$\frac{1}{T} e_{\alpha\beta} \frac{\partial T}{\partial r_\beta} \left[\frac{6}{5} \left(\frac{35}{4} + \frac{T}{\mu} \frac{d\mu}{dT} \right) S_{3/2}^{(1)} - \frac{19}{5} S_{3/2}^{(2)} \right] +$$
$$\frac{6}{5p} e_{\alpha\beta} \frac{\partial p}{\partial r_\beta} S_{3/2}^{(1)} - \frac{6}{5} \frac{\partial e_{\alpha\beta}}{\partial r_\beta} S_{3/2}^{(1)} \Big\} +$$
$$\langle V_\alpha V_\beta \rangle \frac{\mu^2}{p^2} \left\{ \frac{2}{3} \left(\frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right) \frac{\rho}{p} e_{\alpha\beta} \operatorname{div} \mathbf{u} + \right.$$
$$\frac{\rho}{p} \left(\frac{D_0}{Dt} e_{\alpha\beta} - 2 \frac{\partial u_\gamma}{\partial r_\alpha} e_{\gamma\beta} \right) + \frac{\rho}{p} e_{\alpha\gamma} e_{\gamma\beta} \left(4 - \frac{40}{49} S_{5/2}^{(1)} \right) +$$
$$\frac{1}{T} \frac{\partial^2 T}{\partial r_\alpha \partial r_\beta} \left(\frac{3}{2} - \frac{9}{7} S_{5/2}^{(1)} \right) + \frac{9}{7pT} \frac{\partial T}{\partial r_\alpha} \frac{\partial p}{\partial r_\beta} S_{5/2}^{(1)} +$$
$$\frac{1}{T^2} \frac{\partial T}{\partial r_\alpha} \frac{\partial T}{\partial r_\beta} \left[\frac{3}{2} \frac{T}{\mu} \frac{d\mu}{dT} - \frac{9}{7} \left(2 + \frac{T}{\mu} \frac{d\mu}{dT} \right) S_{5/2}^{(1)} + \right.$$
$$\left. \left. \frac{35}{24} \left(1 - \frac{27}{70} \frac{A_4}{A_2} \right) \left(1 + \frac{1}{4} \frac{A_4}{A_2} \right)^{-1} S_{5/2}^{(2)} \right] \right\} +$$
$$\langle V_\alpha V_\beta V_\gamma \rangle \frac{\rho \mu^2}{p^3} \left\{ \frac{1}{T} e_{\alpha\beta} \frac{\partial T}{\partial r_\gamma} \left[\frac{2}{3} \left(\frac{5}{2} + \frac{T}{\mu} \frac{d\mu}{dT} \right) - \right. \right.$$
$$\left. \left. \left(3 - \frac{5}{16} \frac{A_4}{A_2} \right) \left(1 + \frac{5}{12} \frac{A_4}{A_2} \right)^{-1} S_{7/2}^{(1)} \right] - \frac{2}{3p} e_{\alpha\beta} \frac{\partial p}{\partial r_\gamma} + \frac{2}{3} \frac{\partial e_{\alpha\beta}}{\partial r_\gamma} \right\} +$$
$$\langle V_\alpha V_\beta V_\gamma V_\delta \rangle e_{\alpha\beta} e_{\gamma\delta} \frac{\rho^2 \mu^2}{p^4} \frac{25}{14} \left(1 - \frac{7}{15} \frac{A_4}{A_2} \right) \left(1 + \frac{5}{6} \frac{A_4}{A_2} \right)^{-1}$$

for which conditions (1.2) are automatically satisfied.

It is necessary to supplement (1.6) by the general solution of the homogeneous equation $a_0 + \mathbf{a}\mathbf{V} + a_4 V^2$. However it follows from (1.2) that the coefficients a_0, and a_4 are zero, hence (1.6) is the general solution of (1.1) that satisfies conditions (1.2).

Thus $f^{(2)}$ consists of five groups of terms which with respect to the variable $\mathbf{V}$ represent a scalar, a vector, and symmetric irreducible tensors of the second, third, and fourth rank.

Published formulas contain only groups of terms in the form of a vector and a second rank tensor that define Barnett contributions to the stress tensor and heat flux. However the supplementary terms in $f^{(2)}$ are necessary, for instance, for deriving boundary conditions for Barnett equations and for calculating the super-Barnett contributions to the stress tensor and heat flux and, also, third and fourth order moments of the distribution function.

2. **Super-Barnett corrections to the stress tensor and heat flux.** Using the derived distribution function $f^{(2)}$ it is possible to calculate $p_{\alpha\beta}^{(3)}$ and $\mathbf{q}^{(3)}$, the super-Barnett constributions to the stress tensor and heat flux without determining the distribution function $f^{(3)} = f^{(0)}\Phi^{(3)}$ similarly to the derivation of $p_{\alpha\beta}^{(2)}$ and $\mathbf{q}^{(2)}$ using $f^{(1)}$ [2].

The successive use of the supplementary condition for $f^{(3)}$, the integral equation for $B_{\alpha\beta}$, and the symmetry properties of the collision integral [2], yields

$$p_{\alpha\beta}^{(3)} = m\int V_\alpha V_\beta f^{(3)} d\mathbf{v} = 2kT\int \Phi^{(3)} f^{(0)} \langle W_\alpha W_\beta\rangle\, d\mathbf{v} = kT\int \Phi^{(3)} L(B_{\alpha\beta})\, d\mathbf{v} = kT\int B_{\alpha\beta} L(\Phi^{(3)})\, d\mathbf{v} \tag{2.1}$$

Eliminating $\Phi^{(3)}$ using the equation which satisfies it, we obtain

$$p_{\alpha\beta}^{(3)} = -\frac{\mu}{p}\left(\frac{\partial_1}{\partial t} p_{\alpha\beta}^{(1)} + \frac{D_0}{Dt} p_{\alpha\beta}^{(2)} + p_{\alpha\beta}^{(2)} \operatorname{div}\mathbf{u} + 2\left\langle \frac{\partial u_\alpha}{\partial r_\gamma} p_{\gamma\beta}^{(2)}\right\rangle + \frac{\partial}{\partial r_\gamma} P_{\alpha\beta\gamma}^{(2)} + \frac{4}{5}\left\langle \frac{\partial q_\alpha^{(2)}}{\partial r_\beta}\right\rangle\right) \tag{2.2}$$

where

$$P_{\alpha\beta\gamma}^{(2)} = m\int \langle V_\alpha V_\beta V_\gamma\rangle f^{(2)} d\mathbf{v} = \frac{4\mu^2}{\rho}\left[\left\langle \frac{\partial e_{\alpha\beta}}{\partial r_\gamma}\right\rangle + \left(\frac{5}{2} + \frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{1}{T}\left\langle e_{\alpha\beta}\frac{\partial T}{\partial r_\gamma}\right\rangle - \frac{1}{p}\left\langle e_{\alpha\beta}\frac{\partial p}{\partial r_\gamma}\right\rangle\right] \tag{2.3}$$

is the Barnett approximation for the third order moment of the distribution function.

For the heat flux we similarly have

$$q_\alpha^{(3)} = \frac{m}{2}\int V_\alpha V^2 f^{(3)} d\mathbf{v} = -kT\int \Phi^{(3)} f^{(0)}\left(\frac{5}{2} - W^2\right) V_\alpha d\mathbf{v} = kT\int \Phi^{(3)} L(A_\alpha)\, d\mathbf{v} = kT\int A_\alpha L(\Phi^{(3)})\, d\mathbf{v} \tag{2.4}$$

The supplementary condition on $f^{(3)}$, the equation that defines A_α, and the symmetry properties of the collision integral have been successively used here. Eliminating again $\Phi^{(3)}$ we obtain

$$q_\alpha^{(3)} = -\frac{3\mu}{2p}\left[\frac{\partial_1}{\partial t} q_\alpha^{(1)} + \frac{D_0}{Dt} q_\alpha^{(2)} + \frac{10}{3} q_\alpha^{(2)} \operatorname{div}\mathbf{u} + \frac{\partial u_\alpha}{\partial r_\beta} q_\beta^{(2)} + \frac{4}{5} e_{\alpha\beta} q_\beta^{(2)} - \frac{5}{2}\frac{p}{\rho}\frac{\partial}{\partial r_\beta} p_{\alpha\beta}^{(2)} - \frac{1}{\rho} p_{\alpha\beta}^{(1)} \frac{\partial}{\partial r_\gamma} \times p_{\beta\gamma}^{(1)} - \frac{1}{\rho} p_{\alpha\beta}^{(2)} \frac{\partial p}{\partial r_\beta} + P_{\alpha\beta\gamma}^{(2)} e_{\gamma\beta} + \frac{1}{2}\frac{\partial}{\partial r_\beta}\left(P_{2|\alpha\beta}^{(2)} + \frac{1}{3} P_{4|}^{(2)}\delta_{\alpha\beta}\right)\right] \tag{2.5}$$

$$P_{2|\alpha\beta}^{(2)} = m\int \langle V_\alpha V_\beta\rangle V^2 f^{(2)} d\mathbf{v} = \frac{p\mu^2}{\rho^2}\left[\frac{28}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{\rho}{p} e_{\alpha\beta} \operatorname{div}\mathbf{u} + 14\frac{\rho}{p}\left(\frac{D_0}{Dt} e_{\alpha\beta} - 2\left\langle \frac{\partial u_\gamma}{\partial r_\alpha} e_{\gamma\beta}\right\rangle\right) + \frac{472}{7}\frac{\rho}{p}\langle e_{\alpha\gamma} e_{\gamma\beta}\rangle + \frac{39}{T}\left\langle \frac{\partial^2 T}{\partial r_\alpha \partial r_\beta}\right\rangle - \frac{18}{pT}\left\langle \frac{\partial T}{\partial r_\alpha}\frac{\partial p}{\partial r_\beta}\right\rangle + 39\left(\frac{12}{13} + \frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{1}{T^2}\left\langle \frac{\partial T}{\partial r_\alpha}\frac{\partial T}{\partial r_\beta}\right\rangle\right]$$

$$P_{4|}^{(2)} = m\int V^4 f^{(2)} d\mathbf{v} = \frac{p\mu^2}{\rho^2}\left[20\frac{\rho}{p} e_{\alpha\beta} e_{\beta\alpha} + \frac{45}{T}\frac{\partial^2 T}{\partial r_\alpha^2} - \right.$$

$$\frac{45}{pT}\frac{\partial T}{\partial r_\alpha}\frac{\partial p}{\partial r_\alpha}+\frac{45}{T^2}\left(\frac{7}{2}+\frac{T}{\mu}\frac{d\mu}{dT}\right)\frac{\partial T}{\partial r_\alpha}\frac{\partial T}{\partial r_\alpha}\bigg]$$

The Maxwell method was used in [1] for obtaining higher approximations for the stress tensor and the heat flux. Formulas obtained in [1] are transformed into (2.2)(2.3), and (2.5) by expanding the derivative $\partial/\partial t$ in series in Knudsen number Kn (as in the Chapman-Enskog method) and retaining terms of order Kn^2. However the respective formulas for $P_{2|\alpha\beta}$ and $P_{4|}$ contain errors in [1]. The source of errors there is formula (8.10) whose correct form is

$$-\frac{p}{\rho\mu}\left(\rho P_{4|\alpha}-\frac{28}{3}ph_\alpha+\frac{2}{3}p_{\alpha\beta\gamma}p_{\beta\gamma}+\frac{28}{3}p_{\alpha\beta}h_\beta\right)$$

We would also point out that the expressions for $f^{(2)}$, $q_\alpha^{(3)}$, $P_{2|\alpha\beta}^{(2)}$, and $P_{4|}^{(2)}$, in the author's paper [6] contain errors which have been corrected here.

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REFERENCES

1. Ikenberry, E. and Truesdell, C., On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory, I. J. Rational and Mech. Analysis, Vol. 5, No. 1, 1956.

2. Chapman, S. and Cowling, T., Mathematical Theory of Inhomogeneous Gases, Cambridge University Press, 1960.

3. Val'dman, L., Transport phenomena in gases under average pressure, book: Thermodynamics of Gases. Moscow, "Mashinostroenie", 1970.

4. Burnett, D., The distribution of molecular velocities and the mean motion in a nonuniform gas. Proc. London Math. Soc., Vol. 40, No. 5, 1935.

5. Lun'kin, Iu. P., Boundary layer equations and their boundary conditions in the case of motion in a slightly rarefied gas at supersonic velocities. PMM, Vol. 21, No. 5, 1957.

6. Shavaliev, M. Sh., Certain results relative to the Barnett and super-Barnett approximations. Trans. 4-th All Union Conf. on Dynamics of Rarefied Gas and Molecular Gasdynamics. Moscow. Izd. TsAGI, 1977.

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